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FIXED POINT RESULTS FOR EXPANSION MAPS IN PARITAL-2-METRIC SPACE

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ABSTRACT
In this work, some novel fixed point theorems for generalized expansion maps in framework of partial-2-Metric spaces are investigated. These theorems generalise numerous previous results from literature.

Keywords: Metric space, Expansion mapping, Fixed points.

INTRODUCTION

In 1886, French Mathematician Henri Poincare [1] began by experimenting with the notion of fixed point. His result assures the existence of at least one fixed point for an orientation and area-conserving twisted homomorphism of a ring with its boundaries moving in diametrically opposite directions. This result was generated over the movement of astronomical bodies. Later on, Poincare [1] observed that this result does not ensure the existence of an exact solution (but rather an approximate solution). Later on, following his work, L.E.J. Brouwer [2] studied the behaviour of continuous maps in finite-dimensional spaces and the theory was further given extensions by J.Schauder [3] to infinite dimensional spaces. Later on, S. Banach [4] provided the first fixed point solution in the metric framework in 1922 and the renowned result is known as BCP (Banach Contraction Principle).

Gahler proposed the concept of 2-Metric spaces [5] which was a unique structural approach towards metrical theory. Many authors generalized his results in various ways (See [6], [7], [8], [9]). Then, Matthews [16] pioneered partial metric space while working in the realm of computer semantics. An element's self-distance does not have to be zero in this space and a new metric axiom is added to the existing ones: $p(b,b) \leq p(b,c) \forall\, b,c$. These results featured a new direction to research and many authors proved useful results and applications using these theorems. (See [10], [11], [12], [13], [14], [15]) Recently, authors targeted the topological properties of partial metric and 2-Metric spaces to investigate some theorems. The current study comprises of generalised expansion map findings in the context of partial 2-Metric spaces.

PRELIMINARIES

Definition 1.[5] For any set A, the function $\tilde{d} : A \times A \times A \rightarrow \mathbb{R}^+$satisfying the following axioms:
(m1) For $b, l \in A\ (b \neq l)$, there is $w \in A$ such that $d(b,l,w) \neq 0$;
(m2) $\tilde{d}(b,l,w) = 0$ if two of the elements $b,l,w \in A$ are equal;
(m3) $\tilde{d}(b,l,w) = \tilde{d}(b, w, l) = \ldots.$;
(m4) $\tilde{d}(b,l,w) \leq \tilde{d}(b, l, t) + \tilde{d}(b, t, w) + \tilde{d}(b, l, t)$,
for each triplet $b, l, w, t \in A$, is said to be a 2-Metric and $(A, \tilde{d})$ is labelled a 2-Metric space.

Definition 2.[5] For a 2-Metric space $(A, \tilde{d})$, a sequence $\{b_n\}$ converges to $b \in A$ if $lim_{n\rightarrow\infty} \tilde{d}(b_n, b,) = 0$ for all $a \in A$ and we write $lim_{n\rightarrow\infty} b_n = b$.

Definition 3.[5] A sequence $\{b_n\}$ is Cauchy in $(A, \tilde{d})$ if $lim_{m,n\rightarrow\infty} \tilde{d}(b_n, b_m, l) = 0$ for all $l \in A$.

Definition 4.[16] For any set A, the map $\tilde{p} : A \times A \times A \rightarrow \mathbb{R}^+$ is labelled a partial metric if it satisfies the conditions below for every $b, c, w \in A$:
(p1) $b = c$ iff $\tilde{p}(b, b) = \tilde{p}(c, c) = \tilde{p}(b, c)$;
(p2) $\tilde{p}(b, b) \leq \tilde{p}(b, c)$;
(p3) $\tilde{p}(b, c) = \tilde{p}(c, b)$;

(p4) $\tilde{p}\,(b,\,c)\;\le\;\tilde{p}\,(b,\,t)\;+\;\tilde{p}\,(t,\,c)\;-\;\tilde{p}\,(t,\,t)$.
A partial metric space (PMS) is the set A with the metric $\tilde{p}$and denoted by $(A,\,\tilde{p}\,)$.
Example 1.[16] If $A=\mathbb{R}^+$and $\tilde{p}\,(b,c)\;=\;max\{b,c\}$ for each $b,c\;\in\mathbb{R}^+$, then $(A,\tilde{p}\,)$ is a PMS.
Definition 5.[16] Any sequence $\{b_n\}$ in a PMS $(A,\tilde{p}\,)$is

(a) convergent if there is an element b ∈ A for which

$$\tilde{p}(b,b)\;=\;lim_{n\to\infty}\tilde{p}(b_n,b),$$

(b) called Cauchy if $lim_{s,t\to\infty}\tilde{p}(b_s,b_t)$ is finite.
Definition 6.[16] A PMS $(A,\tilde{p})$ is complete if each Cauchy sequence $\{b_n\}$ from A converges in such a way that $\tilde{p}(b,b)\;=\;lim_{s,t\to\infty}\tilde{p}(b_s,b_t)$.
The present work presents some novel results on partial-2-metric space through expansion mappings.

New Findings
Definition 7. For any set A, a map $\varsigma:\,A^3\;\to\mathbb{R}^+$is labelled a partial-2-Metric on A if the conditions written below hold true for each $v,l,w,z\;\in\;A$:
(P_2M1)$\varsigma(v,l,z)\;=\;0$ when two or three of v,l,z coincide;
(P_2M2)$\varsigma(v,l,z)\;=\;\varsigma(v,z,l)\;=\;\varsigma(z,\,v,l)\;=\;\ldots;$
(P_2M3)$\varsigma(v,v,v)\;\le\;\varsigma(v,l,z)$;
(P_2M4)$\varsigma(v,l,z)\le\;\varsigma(v,l,w)+\;\varsigma(v,w\,z)+\;\varsigma(w,l,\,z)-\;\varsigma(w,w,w)$.
The duo $(A,\,\varsigma)$ is then called a Partial-2-Metric (P_2M) space.
Example 2. Let a function $\varsigma:\,A^3\;\to\mathbb{R}^+$ be defined by

$$\varsigma(v,l,z)\;=\;min\,\{|v-l|,\;|l-z|,\;|z\;-\;v|\}.$$

Then ς is a partial-2-Metric.
Remark 2. Every 2-Metric space is a P_2M space as by (m2), $\varsigma(v,v,v)=\;0\;\le\;\varsigma(v,l,z)$ and (P_2M4) is satisfied by using (m2) and (m4).
The above example demonstrates that the opposite does not hold true.
Lemma 1. Let $(A,\,\varsigma)$ be a (P_2M) space and d_ς: $A^3\;\to\;[0,\,\infty)$ be defined as

$$d_\varsigma(v,l,w)\;=\;3\varsigma(v,l,w)\;-\;\varsigma(v,v,v)\;-\;\varsigma(l,l,l)\;-\;\varsigma(w,w,w),$$

then (A,d_ς) is a 2-Metric space.
Definition 8. Let $(A,\,\varsigma)$ be a P_2M space and $\varsigma:A^3\;\to\;[0,\,\infty)$ be the corresponding partial-2-Metric defined on A.
The map $h:\;A\;\to\;A$ is an expansion map if there is a constant $\beta\;\in\;(1,\infty)$ such that
(3.1) $\varsigma(hv,hl,t)\;\ge\;\beta\varsigma(v,l,t)$ for $v,l,t\;\in\;A$.
Lemma 2. Let $(A,\,\varsigma)$ be a P_2M space and $\{v_r\}$be a sequence in A such that
(3.2) $\varsigma(v_{r+1},v_r,\,t)\;\le\;\beta\varsigma(v_r,v_{r-1},\,t)$;
for fixed $t\;\in\;A,\beta\;\in\;(0,\,1)$and $r\;=\;1,\,2,\;\ldots$. Then $\{v_r\}$ is a Cauchy sequence in A.
Proof. By induction, we get for fixed $t\;\in\;A$,

$$\varsigma(v_{r+1},v_r,t)\;\le\;\beta\varsigma(v_r,v_{r-1},t)$$
$$\le\;\beta^2\,\varsigma(v_{r-1},v,t)$$

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(3.3) $\le\;\beta^k\,\varsigma(v_1,v_0,t)$
Also, $\max\{\varsigma(v_r,v_r,t),\varsigma(v_{r+1},v_{r+1},t)\}\le\;\varsigma(v_r,v_{r+1},t)$.

From (3.3),
(3.4) $max\{\varsigma(v_r,v_r,t),\varsigma(v_{r+1},v_{r+1},t)\}\;\le\;\beta^k\,\varsigma(v_1,v_0,t)$
Therefore, by Lemma 3.4,
$\varsigma(v_r,v_{r+1},t)=\;3\varsigma(v_r,v_{r+1},t)-\;\varsigma(v_r,v_r,v_r)$

$$- \varsigma(v_{r+1}, v_{r+1}, v_{r+1}) - \varsigma(t,t,t)$$
$$\leq 3\varsigma(v_r, v_{r+1}, t) + \varsigma(v_r, v_r, v_r) + \varsigma(v_{r+1}, v_{r+1}, v_{r+1})$$
$$+ \varsigma(t,t,t)$$
$$\leq 3\varsigma(v_r, v_{r+1}, t) + \varsigma(v_r, v_{r+1}, t) + \varsigma(v_r, v_{r+1}, t)$$
$$+ \varsigma(v_r, v_{r+1}, t)$$
$$= 6\varsigma(v_r, v_{r+1}, t)$$
$$\leq 6\beta\, k\, \varsigma(v_r, v_{r+1}, t) \quad \text{(by 3.3)}$$

This shows that $lim_{r\to\infty}\varsigma(v_r, v_{r+1}, t) = 0$.
Therefore, $\{v_r\}(r \in \mathbb{N})$is Cauchy in A .

Theorem 1. Let (A, ς) be a P_2M space equipping completeness and h be an onto self mapping on A satisfying the expansive condition

(3.5) $\varsigma(hv, ha, t) \geq \alpha\varsigma(v,a,t) + \beta\varsigma(v, hv, t) + \gamma\varsigma(a, ha, t)$

for $v, a, t \in A$, $\alpha, \beta, \gamma \in (1, \infty)$ with $\alpha + \beta + \gamma > 1$ and for fixed $t \in A$.
Then f has one fixed element.

Proof. Let $v_0 \in A$. As A is onto, choose $v_1 \in A$ for which $hv_1 = v_0$. Continuing the same process, there is a sequence $\{v_r\}$ $(r \in \mathbb{N})$ such that $v_{r-1} = hv_r$; $r = 1, 2, \ldots$
Assume that$v_r \neq v_{r-1}$ $\forall\, r = 1, 2, \ldots$.
For some fixed $w \in U$,
It follows by (3.5),

$$\varsigma(v_{r-1}, v_r, t) = \varsigma(hv_r, hv_{r+1}, t)$$
$$\geq \alpha\varsigma(v_r, v_{r+1}, t) + \beta\varsigma(v_r, hv_r, t)$$
$$+ \gamma\varsigma(v_{r+1}, hv_{r+1}, t)$$
$$= \alpha\varsigma(v_r, v_{r+1}, t) + \beta\varsigma(v_r, v_{r-1}, t)$$
$$+ \gamma\varsigma(v_{r+1}, v_r, t)$$
$$\Rightarrow (1-\beta)\varsigma(v_r, v_{r-1}, t) \geq (\alpha + \gamma)\varsigma(v_r, v_{r+1}, t)$$
$$\Rightarrow \varsigma(v_r, v_{r+1}, t) \leq \left(\frac{1-\beta}{\alpha + \gamma}\right) \varsigma(v_r, v_{r-1}, t)$$

By Lemma 3.6, $\{v_r\}$ $(r \in \mathbb{N})$is Cauchy in A. As (A, ς) is complete, $\{v_r\}$ $(r \in \mathbb{N})$converges in A.
Let $v^* \in A$ such that $\lim_{r\to\infty} v_r = v^*$.
Correspondingly, $\exists\, v \in A$ with $hv = v^*$. Now,

$$\varsigma(v_r, v, t) = \varsigma(hv_{r+1}, hv, t)$$
$$\geq \alpha\varsigma(v_{r+1}, v, t) + \beta\varsigma(v_{r+1}, hv_{r+1}, t)$$
$$+ \gamma\varsigma(v, hv, t)$$

As $r \to \infty$, using (P_2M1), it follows

$$0 = \varsigma(v^*, v^*, t) \geq \alpha\varsigma(fh, v, t) + \beta(0) + \gamma\varsigma(v, hv^*, t)$$
$$\Rightarrow 0 \geq (\alpha + \gamma)\varsigma(hv^*, v, t)$$

which implies $\varsigma(hv^*, v, t) = 0$.
Thus, h$(v) = v$.
For uniqueness, let $l \in A$ such that $h(l) = l$. Now,

$$\varsigma(v, l, t) = \varsigma(hv, hl, t)$$
$$\geq \alpha\varsigma(v, l, t) + \beta(v, hv, t) + \gamma\varsigma(l, hl, t)$$
$$\Rightarrow \varsigma(v, l, t) \geq \alpha\varsigma(v, l, t)$$

$> \varsigma(v, l, t)$ which is a contradiction.
So, we must have $v = l$.
Hence the uniqueness.

Remark 3. By putting $\beta = \gamma = 0$ in Theorem 1, the following outcome is obtained:

Corollary 1. Let (A, ς) be a P_2M complete space and h be an onto map on A satisfying the expansive condition

$$\varsigma(hv, hl, t) \geq \alpha\varsigma(v, l, t)$$

for $v, l \in A$, with $\alpha > 1$ and fixed $t \in A$. Then h has one fixed element.

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